EQUATIONS OF MOTION OF THE MATERIAL IN A SHAKER

A. F. Ryzhkov and A. P. Baskakov

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The behavior of a layer of coarse material in a vertically vibrating vessel is analyzed. The equations of motion and the initial conditions are found for the interval at which the material is tossed from the bottom of the vessel, with the material treated as a loose medium. The results are compared with experiment.

The disperse material in a shaker or vertically vibrating vessel is periodically tossed into flight from the bottom. Kroll reported the first attempts to calculate this motion [1]; further work was reported by Josida and Kausaka [2]. Assuming an absolutely rigid porous medium not touching the walls of the vessel, these investigators found the trajectory for this particular model, using as initial conditions an equation for the separation of a heavy mass point. However, experimental results show that the equations derived here are accurate only if the depth of the material in the shaker is small [1, 3].

An oscilloscopic study has been made of the pressure exerted by a coarse material (synthetic corundum with an average particle size of 1.32 mm) on a membrane strain gauge embedded in the wall by the procedure described in [4]. The results show that the material in the shaker exerts a pressure on the wall throughout the vibration period. The magnitude of this pressure during the flight stage is governed primarily by the direction in which the material is moving with respect to the vessel. This pressure changes abruptly when the particles stop rising and begin to fall. During the contact stage the pressure is maximal near the bottom of the vessel, while during the flight stage the stress is removed from the lower part of the material, and compressional stress is retained only in the middle of the material.

The pressure exerted by the material on the wall generates a friction force between the wall and the material; this friction retards the motion of the material. The vibration of a coarse material is known to involve a close packing of particles, because of the smallness of the hydrodynamic forces which arise during the periodic expulsion of gas from the pulsating volume below the material [2, 4]. Special observations of the rate of particle mixing in the layer of material have shown that the velocity of the vibrational-convective particle motion is one or two orders of magnitude lower than the vibration velocity. Accordingly, in a study of the vertical motion of the material the relative motion of the particles in the layer of material can be neglected in comparison with the height to which the material is tossed, and the material in the flight stage can be treated as a rigid body whose motion changes under the influence of gravitation and the dry friction with the wall of the vessel:

$$\frac{d^2u}{dt^2} = -g - \mathrm{sign}(v)\, f_{\mathrm{T}}, \tag{1}$$

where

$$\mathrm{sign}(v) = \begin{cases} +1 & \text{for} \quad v > 0, \\ -1 & \text{for} \quad v < 0. \end{cases} \tag{2}$$

The limiting value of the dry-friction force mf_{T} is related to the compressional stress in the cylindrical layer by

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$$f_{\text{т}} = \frac{1}{m}\int\limits_F \tau_F\, dF = \frac{4\alpha\lambda}{\rho_{\text{м}}(1-\varepsilon)\, x_0}\int\limits_0^{x_0} \sigma_x dx. \tag{3}$$

We will determine the stresses within the material in several steps. The stresses existing in the layer during the flight stage are due completely to the stresses retained in the loose material after its separation from the bottom of the vessel.

The accelerated motion of the vessel in the gravitational field leads to elastic compressional strain at the lower boundary of the material layer; this strain is accompanied by an irreversible slipping of adjacent particles at contact points, which tends to attenuate a signal propagating into the interior of a loose medium. Since the signal is transmitted essentially instantaneously along the material [5], we can describe the inertial acceleration j of the particles in the layer by the following equation, if we assume an exponential damping:

$$j = j_0 \exp\left[-\eta\left(x_0 - x\right)\right], \tag{4}$$

where $j_0 = A_0\omega^2 \sin\omega t$ is the vibration acceleration.

The vertical compression of the material and the related tendency of the layer to expand in the horizontal direction leads to normal and tangential stresses at the walls of the vessel. The results is dry friction between the walls and the fixed material, which has an unloading effect on the material. During this stage the stress state of the material is determined from the force-balance condition

$$\frac{d\sigma_x}{dx} = \rho_{\text{м}}(1-\varepsilon)(g+j) - 4\tau_F/D. \tag{5}$$

Integrating (5), using $\tau_F = \alpha\lambda\sigma_x$ and (4), and substituting into the solution the dynamic separation condition reflecting the vanishing of the reaction of the bottom, $\sigma_x(x_0) = 0$, we can find the time separation $t_0^{(-)}$ and the stress retained in the medium at this time:

$$\sin\omega t_0^{(-)} = \frac{\varphi_0}{K_0}; \tag{6}$$

$$\sigma_x^{0(-)} = \frac{\gamma}{\varkappa/D}\left[1-\exp\left(-\frac{\varkappa}{D}x\right)\right]\left(1-\frac{K}{K_0}\,\frac{\varphi_0}{\varphi}\right), \tag{7}$$

where

$$K_0 = \frac{A_0\omega^2}{g}; \qquad K = \frac{A\omega^2}{g}; \qquad \varkappa = 4\alpha\lambda; \quad \gamma = \rho_{\text{м}}(1-\varepsilon)\, g; \tag{7'}$$

$$\varphi_0 = \left(1+\frac{\eta}{\varkappa/D}\right)\frac{1-\exp\left(-\frac{\varkappa}{D}x_0\right)}{1-\exp\left[-\left(\eta+\frac{\varkappa}{D}\right)x_0\right]}. \tag{7''}$$

The quantity φ is defined like φ_0, but with x_0 in place of x.

At time $t_0^{(+)}$, when the material begins to rise above the bottom of the vessel, the friction force changes sign ($v > 0$) and, acting in the same direction as the gravitational force, retards the motion of the material, thereby increasing the stress in the material. Since the flight stage is very brief ($\sim 10^{-2}$–10^{-3} sec) we can assume that the packing of particles at the time of separation is retained without any significant change throughout the flight; then the stresses in the tossed material in the case $v > 0$ are given by Eq. (5), where the sign of τ_F is changed. The solution of this equation yields

$$\sigma_x^{0(+)} = \frac{\gamma}{\varkappa/D}\left[\exp\left(\frac{\varkappa}{D}x\right)-1\right]\left(1-\frac{K}{K_0}\,\frac{\varphi_0}{\varphi^{(+)}}\right), \tag{8}$$

where

$$\varphi^{(+)} = \left(1+\frac{\eta}{\varkappa/D}\right)\frac{\exp\frac{\varkappa}{D}x-1}{\exp\left(\frac{\varkappa}{D}-\eta\right)x-1}. \tag{9}$$

When the sign of τ_F changes again, as the material falls to the bottom ($v < 0$), the stress in the material again becomes equal to the value $\sigma_x^{0(-)}$ given by Eq. (7).

Accordingly, the term taking into account the external friction in equation of motion (1) is written

$$f_T = \begin{cases} (\psi_{(+)} - 1)\, g & \text{for} \quad v > 0, \\ (1 - \psi_{(-)})\, g & \text{for} \quad v < 0, \end{cases} \tag{10}$$

where the coefficients $\psi_{(+)}$ and $\psi_{(-)}$ are governed by the mechanical and geometric properties of the loose material and do not depend explicitly on the vibration characteristics:

$$\psi_{(+)} = \frac{1}{x_0}\,\frac{D}{\varkappa}\left(\exp\frac{\varkappa}{D}x - 1\right)\left\{1 - \frac{\varphi_0 \exp(-\eta x_0)}{1 + \frac{\eta}{\varkappa/D}}\left[1 - \frac{\varkappa(\exp\eta x_0 - 1)}{D\eta\left(\exp\frac{\varkappa}{D}x_0 - 1\right)}\right]\right\} \tag{11}$$

and

$$\psi_{(-)} = \frac{1}{x_0}\,\frac{D}{\varkappa}\left[1 - \exp\left(-\frac{\varkappa}{D}x_0\right)\right]\left\{1 - \frac{\varphi_0 \exp(-\eta x_0)}{1 + \frac{\eta}{\varkappa/D}}\left[1 - \frac{\varkappa(1 - \exp(-\eta x_0))}{D\eta\left(1 - \exp\left(-\frac{\varkappa}{D}x_0\right)\right)}\right]\right\}. \tag{12}$$

Using Eqs. (2)-(12), we can write the equation of motion for this model, Eq. (1), as

$$\frac{d^2s}{dt^2} + \frac{d^2l}{dt^2} = -\begin{cases} \psi_{(+)}\, g & \text{for} \quad v > 0, \\ \psi_{(-)}\, g & \text{for} \quad v < 0. \end{cases} \tag{13}$$

where s is the height to which the material is tossed above the bottom of the vessel, which itself is moving according to $l = A_0 \sin \omega t$; $s + l = u$.

This equation describes the motion of a loose material in a steady-state body-force field proportional to the gravitational force, with an abrupt change is the magnitude of this force when the direction of the relative velocity changes. The difference between the equations of motion found on the basis of this model and those found on the basis of the Kroll model is governed by the magnitude of the coefficients $\psi_{(+)}$ and $\psi_{(-)}$ and by the time over which they are effective.

For small heights, with $x_0 \ll 1/\eta$ and $x_0 \ll D/\varkappa$, and for which the external friction force is extremely slight, we have $\psi_{(+)} = \psi_{(-)} \approx 1$, so that the layer moves along the trajectory of a mass point in a gravitational force field. As the looseness becomes more important, the friction force increases, and the motion of the layer is changed. As the material moves upward ($v > 0$) the friction force retards the ascent, and as the material moves downward the friction force retards the descent.

The increase of $\psi_{(+)}$ with increasing x_0 is unbounded. When the body forces acting on the material after the reaction of the bottom of the vessel vanishes reach the level of the inertial vibration forces ($\psi_{(+)} = K_0$), the material, "pinched" by the walls, begins to move along with the vessel and does not leave the bottom.

At heights x_0 corresponding to $\psi_{(+)} < K_0$, the material is tossed above the bottom. However, the flight stage begins at the time $t_0^{(-)}$ at which the stress at the lower boundary of the layer vanishes, given in (6), only under the condition $\psi_{(+)} \leq \varphi_0$. Otherwise, the flight stage begins later, at a time given by

$$\sin \omega t_0' = \frac{\psi_{(+)}}{K_0}. \tag{14}$$

Over the time interval $t_0^{(-)}$-t_0' the material moves along with the vessel walls and exerts no pressure on the bottom of the vessel.

On the other hand, the quantity $\psi_{(-)}$ decreases from one to zero as x_0 increases; accordingly, the deeper the layer, the slower the descent of the material.

Accordingly, we can describe the motion of the material in the shaker during the ascent and descent in dimensionless form by equations

$$\frac{d^2\bar{s}_1}{d\theta^2} = \sin\theta - \frac{\psi_{(+)}}{K_0} \quad \text{for} \quad \theta' < \theta < \theta'', \tag{15}$$

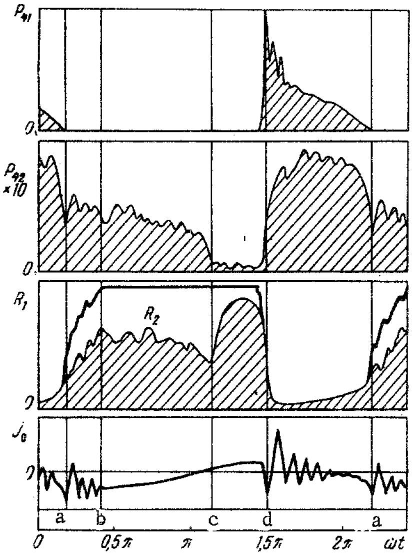

Fig. 1

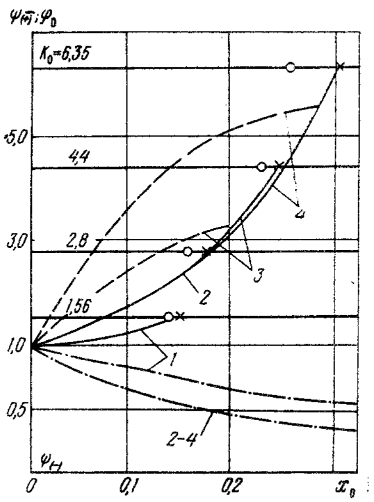

Fig. 2

Fig. 1. Behavior of the particle pressure on the bottom of the vessel (P_{41}) and on the walls of the vessel (P_{42}), the electrical resistance between the material and the bottom (R_1), and the resistance between the walls of the vessel (R_2) recorded during the vibration of a 160-mm layer of graphite at f = 16 Hz and A_0 = 2.73 mm. Here j_0 is the signal from the piezoelectric transducer.

Fig. 2. The coefficient φ_0 (dashed curves), $\psi_{(+)}$ (solid curves), and $\psi_{(-)}$ (dot-dash curves) calculated from Eqs. (7"), (11), and (12) as functions of x_0 (in m) for $\varkappa / D = 10\ m^{-1}$, D = 0.1 m, and the following values of K_0 and η, respectively: 1) 1.56, 2.2; 2) 2.82, 13; 3) 4.4, 28; 4) 6.35, 50.

$$\bar{s}_1(\theta') = \frac{d\bar{s}_1}{d\theta}(\theta') = 0, \tag{15'}$$

$$\frac{d^2\bar{s}_2}{d\theta^2} = \sin\theta - \frac{\psi_{(-)}}{K_0} \quad \text{for} \quad 0'' < \theta \leqslant \theta_d, \tag{16}$$

$$\bar{s}_2(\theta'') = \bar{s}_1(\theta''), \tag{16'}$$

where $s_{1,2} = s / A_0$; $\theta = \omega t$; and θ_d is the phase angle of the descent.

The dimensionless time θ' at which the flight begins is to be found as a function of the relation between φ_0 and $\psi_{(+)}$ from Eq. (6) or (14). The phase angle θ'' at which the direction of velocity v changes is found from the condition

$$\frac{d\bar{s}_1}{d\theta}(\theta'') = 0; \qquad \frac{d^2\bar{s}_1}{d\theta^2}(\theta'') = 0. \tag{17}$$

In an experimental check of this model we studied a graphite layer with particle sizes of 0.25-1.0 mm in a Plexiglas column 100 mm in diameter with a layer depth of x_0 = 50-600 mm. The vibration frequency was varied from 10 to 30 Hz with a vibration amplitude of A_0 = 2.73 mm. In the experiments we measured the instantaneous electrical resistance of the material and determined whether there was electrical contact between the material and the bottom of the vessel. Simultaneously, on the basis of the appearance and disappearance of a load on a membrane strain gauge at the bottom of the vessel, we determined the phase angles of the separation, $\omega t_0^{(-)}$, and descent ωt_d, of the material. A ceramic piezoelectric transducer mounted in the bottom was used along with a PDU-1M amplifier to detect the vibration acceleration and the mechanical impacts of the particles. All the changes were recorded on motion-picture film with a loop oscillograph [4].

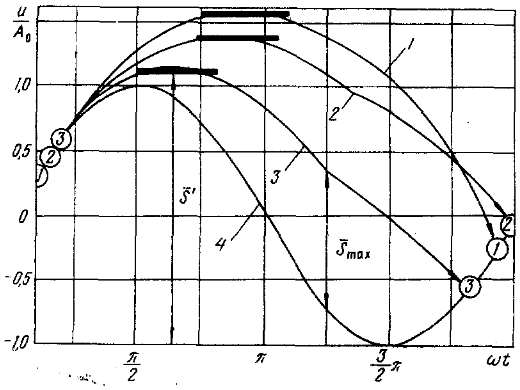

Fig. 3. Trajectories for layers with depths of 0.007 m (1), 0.04 m (2), and 0.08 m (3) according to Eqs. (16) and (17) for the value $K_0 = 2.82$, according to the calculations in [1-3]; 4) trajectory for the bottom of the vessel.

The results (Fig. 1) show that after the pressure exerted by the particles on the strain gauge disappears the electrical resistance between the bottom of the vessel and the layer does not reach its limiting value instantaneously and it displays some fluctuation. At these times the piezoelectric transducer indicates mechanical impacts. At a phase angle of about $\pi/2$ the signals indicating contact of the material with the bottom are cut off, and they do not reappear until the layer reaches the bottom (d in Fig. 1). Accordingly, over the interval ab the layer is at rest with respect to the bottom and near it; it exerts no pressure on the bottom and does not separate from it. This behavior was observed during the shaking of layers more than 50-100 mm deep. At a lower depth the electrical contact between the layer and the bottom is lost when the layer ceases to exert pressure on the bottom. These results support the argument that the tossing of deep layers is retarded by "pinching" by the vessel walls.

The time dependence of the electrical resistance of the material (corresponding to density oscillations of discrete phase and internal stresses in the layer) agrees well with the time dependence of the pressure exerted by the particles on the vessel wall during the shaking. During the contact stage, when the pressure exerted by the particles on the walls is maximal, the electrical resistance of the material reaches its minimum value. When the reaction of the bottom disappears, the stresses in the layer decrease, and the electrical resistance increases slightly. After the direction of motion changes, the resistance increases abruptly, and the layer exerts a minimal pressure on the walls. The abrupt changes in the electrical resistance of the tossed material which occur upon the change of direction demonstrate the validity of the two-stage model for the motion. Theory and experiment can be compared quantitatively by comparing the initial conditions and the trajectories of the actual material (corundum with a particle size of 1.32 mm) and the model, whose properties are shown in Fig. 2. The dependence of the damping factor η on K_0, required for the calculations, was obtained from Eqs. (6) and (7") and the experimental data on the phase angles $\omega t_0^{(-)}$ corresponding to the separation of the material. These phase angles were inferred from the instant at which the load disappeared from the membrane strain gauges at the bottoms of vessels, 0.04-0.25 m in diameter. The values of α and λ used in calculating η were assumed equal to 0.5 in accordance with [5]. The particle size ensured that there were no appreciable pulsations in the hydrodynamic head ($\Delta P/x_0 < \gamma$) [6], thereby facilitating the comparison with theory.

Figure 2 shows the coefficients φ_0, $\psi_{(+)}$, and $\psi_{(-)}$ calculated from Eqs. (7"), (11), and (12); these results show that the delay in the beginning of flight due to friction with the wall ($\psi_{(+)} > \varphi_0$) is appreciable at small vibration parameters ($K_0 = 1.56$ and 2.82), while at large values of K_0, over nearly the entire range of heights x_0, where $K_0 > \psi_{(+)}$, the beginning of flight is determined from the vanishing of the reaction of the bottom, from (6). The crosses in this figure give the maximum height of the material, at which the external friction force reaches the level corresponding to the vibration inertia ($\psi_{(+)} = K_0$), and the phase angle of the beginning of flight becomes equal to $\pi/2$. In this case the layer no longer escapes from the bottom of the vessel.

The data on the limiting layer height was checked with a KM-6 cathetometer, which permitted highly accurate measurements of the excursion s' between the highest position of the tossed material and the lowest position of the bottom (for the case in which the material falls in the last quarter of the period). The layer heights at which s' becomes smaller than twice the vibration amplitude, $2A_0$, evidence of the attainant of the maximum layer height, are shown by the circles in Fig. 2. These values deviate from the calculated values by no more than 15%.

Using the results calculated for the motion of an object in the field of a constant body force of intensity $1/K_0$ (in dimensionless form), given in [1-3], we can easily calculate the trajectories of an object subjected to a force of intensity $\psi_{(+)}/K_0$ for various values of $\varphi_{(+)}$ and for the initial conditions determined from the curves in Fig. 2 (the left branches in Fig. 3). After finding the point corresponding to $v \equiv d\bar{s}_1/d\theta = 0$; ($\bar{s}$ = max) on these trajectories, we continue to follow the object, but along a different trajectory, corresponding to a force $\varphi_{(-)}/K_0$ (the right branches). Their intersections with the trajectories of the bottom give the values of θ_d. The trajectories shown in this figure for three bodies, corresponding to different layer depths, show that only if the layer depth is comparable to the particle size does the layer behave as a Kroll object, since in this case the internal and external friction forces are negligible (curve 1). On the other hand, an increase in the height x_0 leads, as was mentioned above, to an abrupt change in the nature of the trajectory: its height becomes much lower, with a relative increase in the time of flight.

The final results of this study were also checked experimentally. In experiments carried out under the conditions described above, strain gauges were used to measure the times at which the material separated from and struck the bottom of the vessel (the circles in Fig. 3). In addition, a cathetometer was used to determine the optical thickness s' of the gap between the layer and the bottom (the horizontal bars in Fig. 3). The trajectory elements of the real material determined in this manner agree well with theoretical values.

Accordingly, this model gives a satisfactory description of the motion of a real material tossed in a vertically vibrating apparatus over broad ranges of the parameters K_0 and x_0.

NOTATION

u	is the displacement of the material;
l	is the displacement of the bottom of the vessel;
s	is the height of the tossing above the bottom;
D	is the diameter of the vessel;
x	is the instantaneous height;
x_0	is the height of the material;
A_0	is the vibration amplitude;
t	is the time;
ω	is the angular frequency;
j, g	are the inertial and gravitational accelerations.
m, F	are the mass and lateral surface area of the layer;
σ_x, τ_F	are the normal and tangential stresses averaged over the cross section;
α, λ, η	are the coefficient of dry friction, coefficient of lateral pressure, and damping factor;
ρ_M	is the density of the material;
ε	is the porosity of the layer;
ΔP	is the hydraulic resistance of the layer.

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